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Note

Probability of diameter two for Steinhaus graphs

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Abstract

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A Steinhaus graph is a graph with n vertices whose adjacency matrix $(a_{i,j})$ satisfies the condition that $a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j}$ (mod 2) for each $1 < i < j \leq n$. It is clear that a Steinhaus graph is determined by its first row. In "Almost all Steinhaus graphs have diameter two", J. Graph Theory 16 (1992) 213–219 it is shown that almost all Steinhaus graphs have diameter two. Here we generalize to the case where the jth entry of the first row has probability p_j of being 1. Under reasonable conditions it is shown that the probability measure of the set of Steinhaus graphs with diameter two approaches 1 as the number of vertices in the graph approaches infinity.

1. Introduction

Let $(a_{1,j})_{j=2}^{n}$ be a string of 0's and 1's. We define $a_{i,j}$ for $1<i<j\leq n$ inductively by $a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j}$ (mod 2). We complete $(a_{i,j})_{1<i<j\leq n}$ to an $n \times n$ matrix by

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defining $a_{i,i}=0$ and $a_{i,j}=a_{j,i}$ for $1 \le j < i \le n$. This gives an $n \times n$ symmetric matrix of 0's and 1's with 0's on the diagonal. It is therefore the adjacency matrix for some graph. Any graph defined in this way is called a *Steinhaus graph*. The string $(a_{1,j})_{j=2}^{n}$ is called the *generating string* for the graph. For convenience we label the vertex set of a Steinhaus graph with the first n natural numbers, $\{1,2,3,\ldots,n\}$. It is obvious that there are exactly 2^{n-1} labeled Steinhaus graphs of order n.

In [2–4] the problem of finding the diameter of a Steinhaus graph is investigated. In [3] it is shown that the maximum diameter of a nontrivial Steinhaus graph is roughly $n/2$. Furthermore, in [4] Brigham and Dutton show that almost all Steinhaus graphs have diameter at most three and they conjecture that almost all Steinhaus graphs have diameter two. In [2] Brigham and Dutton's conjecture is proved.

Here we generalize by allowing $\Pr(a_{1,j}=1)$ to depend on the column j. Throughout this paper we fix a nonincreasing sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of numbers between 0 and $\frac{1}{2}$ with the property that $\lim_{n\to\infty}(\sqrt[7]{n}\varepsilon_n^2 - 2\log n)=\infty$. We then let $\{p_j\}_{j=2}^{\infty}$ be a sequence with $\varepsilon_j < p_j < 1-\varepsilon_j$ for each $j>1$ and define $q_j = 1-p_j$. We define the set of labeled Steinhaus graphs with n vertices as a sample space and assign a probability measure by requiring that $\Pr(a_{1,j}=1)=p_j$. The purpose of this paper is to prove

Theorem 1.1. *If* $\Pr(a_{1,j}=1)=p_j$ *for each* $j \ge 2$ *then as* $n\to\infty$, *the probability that a Steinhaus graph on n vertices has diameter two approaches* 1.

and

Theorem 1.2. *If* $\Pr(a_{1,j}=1)=p_j$ *for each* $j \ge 2$ *then as* $n\to\infty$, *the probability that the complement of a Steinhaus graph on n vertices has diameter two approaches* 1.

Corollary 1.3. *If there is a constant* δ *such that* $\delta \le p_j \le 1-\delta$ *for each j, then the probability that a Steinhaus graph on n vertices has diameter two approaches* 1. *(Similarly for the complement of a Steinhaus graph.)*

Proof. Since the constant sequence $\varepsilon_j=\delta/2$ satisfies the required condition, the corollary follows. □

Corollary 1.4. *If* $p_j=\frac{1}{2}$ *for every j then almost all Steinhaus graphs and almost all complements of Steinhaus graphs have diameter two.*

Proof. This follows from Corollary 1.3 using $\delta=\frac{1}{2}$. □

Note that Brigham and Dutton's conjecture, which was proved in [2], is Corollary 1.4. What is new here is that we are allowing the probabilities of 1's on the first row to vary with the column. In fact, the theorems state that the probability of a 1 at the $(1,n)$ position can approach 0 (if slowly enough) as n approaches infinity and still the probability of diameter two approaches 1. Certainly, the conditions on the sequence $\{\varepsilon_j\}$ could be weakened. It is an interesting problem to determine necessary and sufficient conditions on $\{p_j\}$ so that Theorems 1.1 and 1.2 are true.

2. Proof of the theorem

Throughout we fix the sequence $\{\varepsilon_j\}$ with the properties listed in Section 1. We use the number n to represent the number of vertices in a Steinhaus graph under consideration. As stated in the introduction, p_j is assumed to satisfy $\varepsilon_j<p_j<1-\varepsilon_j$ and $q_j=1-p_j$ for $2\le j\le n$, and we assume $\Pr(a_{1,j}=1)=p_j$. For each j we let $m_j=\min\{p_j,q_j\}$ and $M_j=\max\{p_j,q_j\}$. Furthermore, we define $p_{i,j}=\Pr(a_{i,j}=1)$ and $q_{i,j}=1-p_{i,j}$. We will use $a_{i,j}$ without further explanation to represent the entry in row i and column j in the adjacency matrix for a Steinhaus graph. Equivalently, we will think of $a_{i,j}$ as a function from the sample space of Steinhaus graphs with n vertices to the set $\{0,1\}$ which is given by the entry at row i and column j of the adjacency matrix. We say an event E and the function $a_{1,j}$ are *position independent* if for every Steinhaus graph G in E, the Steinhaus graph obtained by changing the value of $a_{1,j}$ in the generating string for G is also in E. It is clear that if E and $a_{1,j}$ are position independent then E and the set of all Steinhaus graphs of order n with $a_{1,j}=x$ are independent for either choice of $x\in\{0,1\}$.

The following is [2, Lemma 1.1].

Lemma 2.1. *Let G and G' be Steinhaus graphs of order n with adjacency matrices $(a_{i,j})$ and $(b_{i,j})$ respectively. Suppose that $a_{1,j}=b_{1,j}$ for $j\ne k$ and $a_{1,k}=1-b_{1,k}$. For any $1\le i<j\le n$ with either $j<k$ or $j-i>k-1$, $a_{i,j}=b_{i,j}$. Also, for any $1\le i<j\le n$ if either $j=k$ or $j-i=k-1$ then $a_{i,j}=1-b_{i,j}$.*

In order to estimate probabilities concerning diameters we need to estimate the probabilities $p_{i,j}$ for each $1\le i<j\le n$.

Lemma 2.2. *Let the event E and function $a_{1,k}$ be position independent and $x\in\{0,1\}$. Then for any $i\le n-k+1$, $m_k\le\Pr(a_{i,k+i-1}=x\mid E)\le M_k$. Also, for any $i<k$, $m_k\le\Pr(a_{i,k}=x\mid E)\le M_k$.*

Proof. Let $\{a_{1,j}\}_{j=2}^{n}$ be a generating string for a Steinhaus graph G. Then $\Pr(G)=\prod_{j=2}^{n}\theta_j$ where $\theta_j=p_j$ if $a_{1,j}=1$ and $\theta_j=q_j$ otherwise for each $1<j\le n$. Also, we let $W(G)=\prod_{2\le j\le n,\, j\ne k}\theta_j$. Then $W(G)$ is the probability of the event consisting of G and the Steinhaus graph obtained by changing the kth entry in the generating string for G. To emphasize that θ_j depends on the graph G, we write $\theta_j(G)$.

Now,

$$\Pr(a_{i,k+i-1}=x\mid E)=\frac{1}{\Pr(E)}\Pr(a_{i,k+i-1}=x \text{ and } E)$$

$$= \frac{1}{\Pr(E)} \sum_{a_{i,k+i-1}=x,\ G\in E} \Pr(G)$$

$$= \frac{1}{\Pr(E)} \sum_{a_{i,k+i-1}=x,\ G\in E} W(G)\theta_k(G)$$

$$\geq \frac{m_k}{\Pr(E)} \sum_{a_{i,k+i-1}=x,\ G\in E} W(G).$$

But $\sum_{a_{i,k+i-1}=x,\ G\in E} W(G)=\Pr(E)$ since changing the value of $a_{1,k}$ changes the value of $a_{i,k+i-1}$ by Lemma 2.1. So, $\Pr(a_{i,k+i-1}=x \mid E)\geq m_k$. By using the inequality $\theta_k(G)\leq M_k$ in the above argument the inequality $\Pr(a_{i,k+i-1}=x \mid E)\leq M_k$ follows. A similar argument shows $m_k\leq \Pr(a_{i,k}=x \mid E)\leq M_k$. □

Corollary 2.3. *For any* $1\leq i<j\leq n$, $m_j\leq \Pr(a_{i,j}=x)\leq M_j$ *and* $m_{j-i+1}\leq \Pr(a_{i,j}=x)\leq M_{j-i+1}$.

Proof. Simply let E be the set of all Steinhaus graphs in Lemma 2.2. □

Lemma 2.4. *Let* $k>1$ *and suppose* E *is an event which can be defined in terms of the values of the functions in the set* $F=\{a_{i,j} \mid i<j<k \text{ or } j-i>k-1\}$. *Then* E *and* $a_{1,k}$ *are position independent.*

Proof. By Lemma 2.1 it follows that the values of the functions in F are not effected by changing the value of $a_{1,k}$. Therefore the event and the function are position independent. □

For each $1\leq i<j\leq n$, we let $Z_{i,j}$ denote the probability that the distance from vertex i to vertex j is greater than 2 in a Steinhaus graph on n vertices. The next few lemmas give upper bounds for $Z_{i,j}$. These upper bounds will then be used to prove Theorem 1.1.

Lemma 2.5. *For any* $1\leq i<j\leq n$, $Z_{i,j}\leq \prod_{t=1}^{\lfloor (n-1)/j\rfloor} (1-m_{(t-1)j+2}m_{tj+1})$.

Proof. We fix $1\leq i<j\leq n$. Let $v_t=tj+1$ for $1\leq t\leq \lfloor (n-1)/j\rfloor$. Clearly, $j<v_t\leq n$ for each t. We show inductively that for each r satisfying $1\leq r\leq \lfloor (n-1)/j\rfloor$ the probability that for every $t\leq r$ either vertex i is not adjacent with vertex v_t or else vertex j is not adjacent with vertex v_t is no more than $\prod_{t=1}^{r} (1-m_{(t-1)j+2}m_{tj+1})$. We start the induction by estimating the probability that both $a_{i,v_1}=1$ and $a_{j,v_1}=1$. First $\Pr(a_{i,v_1}=1)\geq m_{v_1}$ by Corollary 2.3. The event $a_{i,v_1}=1$ and the function $a_{1,2}$ are position independent by Lemma 2.4. So $\Pr(a_{i,v_1}=1 \text{ and } a_{j,v_1}=1)\geq m_2m_{v_1}$. Therefore the probability that either i and v_1 are not adjacent or else j and v_1 are not adjacent is at most $1-m_2m_{v_1}=1-m_2m_{j+1}$, which starts the induction.

To do the induction step, we let E_r be the event that for each $t\leq r$, either i and v_t

are not adjacent or else j and v_t are not adjacent. We assume that $r<\lfloor(n-1)/j\rfloor$ and $\Pr(E_r)\leq\prod_{t=1}^{r}(1-m_{(t-1)j+2}m_{tj+1})$. The set E_r and $a_{1,v_{r+1}}$ are position independent by Lemma 2.4. Therefore $\Pr(a_{i,v_{r+1}}=1\cap E_r)=\Pr(a_{i,v_{r+1}}=1\mid E_r)\Pr(E_r)\geq m_{v_{r+1}}\Pr(E_r)$ by Lemma 2.2. Now let E' be the set of Steinhaus graphs in E_r which have $a_{i,v_{r+1}}=1$. It is clear from Lemma 2.4 that E' and $a_{1,rj+2}$ are position independent. It follows from Lemma 2.2 that $\Pr(a_{j,v_{r+1}}=1\cap E')=\Pr(a_{j,v_{r+1}}=1\mid E')\Pr(E')\geq m_{rj+2}\Pr(E')=m_{v_{r+1}}\Pr(E')$. We have $\Pr(E_{r+1})=\Pr(E_r)-\Pr(a_{i,v_{r+1}}=1\cap a_{j,v_{r+1}}=1\cap E_r)=\Pr(E_r)\times(1-\Pr(a_{i,v_{r+1}}=1\cap a_{j,v_{r+1}}=1\mid E_r))\leq\Pr(E_r)(1-m_{v_{r+1}}m_{v_{r+1}}=\prod_{t=1}^{r+1}(1-m_{(t-1)j+2}m_{tj+1})$. □

Lemma 2.6. *For any* $1\leq i<j\leq n$, $Z_{i,j}\leq\prod_{t=0}^{\lfloor(i-2)/(j-i+1)\rfloor}(1-m_{i-t(j-i+1)}m_{j-t(j-i+1)})$.

Proof. The proof is very much like the proof of Lemma 2.5. We fix $1\leq i<j\leq n$. Let $v_t=t(j-i+1)+1$ and E_r be the event that for any $t\leq r$ either vertex i and vertex v_t are not adjacent or else j and v_t are not adjacent. We show by induction that $\Pr(E_r)\leq\prod_{t=0}^{r}(1-m_{i-t(j-i+1)}m_{j-t(j-i+1)})$. The case $r=0$ follows just as in Lemma 2.5.

Suppose $\Pr(E_r)\leq\prod_{t=0}^{r}(1-m_{i-t(j-i+1)}m_{j-t(j-i+1)})$ for some $r<\lfloor(i-2)/(j-i+1)\rfloor$. First note that E_r and $a_{1,j-(r+1)(j-i+1)}$ are position independent and so $\Pr(a_{(r+1)(j-i+1)+1,j}=1\mid E_r)\geq m_{j-(r+1)(j-i+1)}$ by Lemmas 2.4 and 2.2. As before, we let E' be the set of Steinhaus graphs in E_r such that $a_{(r+1)(j-i+1)+1,j}=1$ and note that E' and $a_{1,i-(r+1)(j-i+1)}$ are position independent. Therefore, $\Pr(a_{(r+1)(j-i+1)+1,i}=1\mid E')\geq m_{i-(r+1)(j-i+1)}$ by Lemmas 2.4 and 2.2. As in the proof of Lemma 2.5, the induction step follows. □

Lemma 2.7. *For* $1\leq i<j\leq n$, $Z_{i,j}\leq\prod_{t=1}^{\lfloor(j-2i-1)/2\rfloor}(1-m_{i+t}m_{j-i-t+1})$.

Proof. The proof proceeds in the same way as the previous two lemmas. We define $v_t=i+t$ and E_r to be the event that for any $t\leq r$ either the vertices i and v_t are not adjacent or else j and v_t are not adjacent. The key facts are that E_r and $a_{1,j-i-r}$ are position independent and that E', the set of Steinhaus graphs in E_r with $a_{v_{r+1},j}=1$, and $a_{1,i+r+1}$ are position independent. □

Lemma 2.8. *For any* $1\leq i<j\leq n$, $Z_{i,j}\leq\prod_{t=1}^{\min\{j-i,i-1\}}(1-m_{i-t+1}m_{j-t+1})$.

Proof. Again this proof is by induction. Using the notation of the previous lemmas, we let $v_t=t$ and define E_r as above. It is easy to check that E_r and $a_{1,j-r}$ are position independent. We let E' be the set of Steinhaus graphs in E_r with $a_{v_{r+1},j}=1$. Then E' and $a_{1,i-r}$ are position independent. The reason for the upper bound of $\min\{j-i,i-1\}$ is that if $j-r>i$ then E_r and $a_{1,i}$ are position dependent and if $i-r>1$ then $a_{1,i-r}$ is defined and position independent with E'. □

Lemma 2.9. *There is a number* N *such that if* $n>N$ *then for any* $1\leq i<j\leq n$, *at least*

one of the conditions hold:

(a) $j \leq 3\sqrt{n}$,

(b) $j \geq \sqrt{n} + 2i$,

(c) $j \leq i(1 + 1/\sqrt[7]{n}) - 2$,

(d) $i \geq \sqrt{n}$ *and* $j - i \geq \sqrt[3]{n}$.

Proof. Suppose that $j > 3\sqrt{n}$ and $i < \sqrt{n}$. Then $j > \sqrt{n} + 2\sqrt{n} > \sqrt{n} + 2i$, which is condition (b).

Next suppose that $j > 3\sqrt{n}$, $i \geq \sqrt{n}$, and $j - i < \sqrt[3]{n}$. Then $j - i < \sqrt[3]{n} = \sqrt{n}/\sqrt[6]{n} \leq i/\sqrt[6]{n} < i/\sqrt[7]{n} - 2$ for n sufficiently large, which is condition (c). □

Lemma 2.10. *There is a sequence* $\{g_n\}_{n=1}^{\infty}$ *with* $g_n \to 0$ *which satisfies the following condition. For each natural number* n, *let* $S = \{x_1, x_2, \ldots, x_k\}$ *and* $T = \{y_1, y_2, \ldots, y_k\}$ *be subsets of* $\{1, 2, 3, \ldots, n\}$ *with exactly* k *elements where* $k \geq \sqrt[7]{n}$. *Then* $0 \leq n^2 \prod_{i=1}^{k} (1 - m_{x_i} m_{y_i}) \leq g_n$.

Proof. We let $g_n = n^2(1 - \varepsilon_n^2)^{\sqrt[7]{n}}$. As $\varepsilon_n \leq m_n$, and $\{\varepsilon_n\}$ is nonincreasing it follows that $0 \leq n^2 \prod_{i=1}^{k} (1 - m_{x_i} m_{y_i}) \leq g_n$. It only remains to show that $g_n \to 0$. By taking log of g_n, it is sufficient to show $\lim_{n\to\infty} (2 \log n + \sqrt[7]{n} \log(1 - \varepsilon_n^2)) = -\infty$. Using the series approximation $\log(1-x) < -x$, it is sufficient to show $\lim_{n\to\infty} (\sqrt[7]{n}\varepsilon_n^2 - 2 \log n) = \infty$. But this is exactly the assumption we are making about the sequence $\{\varepsilon_n\}$. □

Now we can prove Theorem 1.1. Since the only graph with diameter one is the complete graph, it is sufficient to show the set of Steinhaus graphs with diameter less than or equal to two has probability 1. Suppose $1 \leq i < j \leq n$. If Lemma 2.9(a) holds then according to Lemmas 2.5 and 2.10, $Z_{i,j} \leq g_n/n^2$, where g_n is as in Lemma 2.10. If Lemma 2.9(b) holds then according to Lemmas 2.7 and 2.10, $Z_{i,j} \leq g_n/n^2$. If Lemma 2.9(c) holds and $n > 2^7$ then $j \leq i + i/\sqrt[7]{n} - 2 < i + i/\sqrt[7]{n} - 2/\sqrt[7]{n} - 1$, from which it follows that $(i-2)/(j-i+1) \geq \sqrt[7]{n}$. According to Lemmas 2.6 and 2.10, $Z_{i,j} \leq g_n/n^2$. Finally, if Lemma 2.9(d) holds then according to Lemmas 2.8 and 2.10, $Z_{i,j} \leq g_n/n^2$. In any case, $Z_{i,j} \leq g_n/n^2$. Now, the probability that a Steinhaus graph of order n has diameter more than two is bounded above by $\sum_{1 \leq i < j \leq n} Z_{i,j} \leq (n(n-1)/2)(g_n/n^2) \to 0$ as $n \to \infty$. Therefore, the probability that a Steinhaus graph of order n has diameter two approaches 1 as n approaches infinity.

The proof of Theorem 1.2 is essentially the same as the above proof. In the proofs of the lemmas, instead of computing the probability that either i and v_t are not adjacent or that j and v_t are not adjacent one would compute the probability that i and v_t are adjacent or j and v_t are adjacent. Otherwise the proof is identical.

In general, suppose $\Pr(a_{1,j} = 1) = p_j$ for every $j > 2$. An open problem is to determine necessary and sufficient conditions on the sequence $\{p_j\}_{j=2}^{\infty}$ which satisfy Theorem 1.1.

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